

Action Principle for the Classical Dual Electrodynamics¹

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Abstract

The purpose of this paper is to formulate an action principle which allows for the construction of a classical lagrangean including both electric and magnetic currents. The lagrangean is non-local and shown to yield all the expected (local) equations for dual electrodynamics.

One of the oldest open problems in the theory of elementary particles is that of the quantization of the electric charge. Although apparently very simple, this experimental result has not yet found theoretical explanation in the context of the standart model of fundamental interactions.

In 1931, PAM Dirac^[1] found an explanation for such quantization based on the lack of symmetry of Maxwell's equations in what concerns their source terms. The presence of magnetic currents in these equations leads, at the quantal level, to the quantization of the electric and magnetic charges.

Since the pioneer work of Dirac, other solutions to the problem have been proposed in the context of unified theories, as GUT's^[2,3] and Kaluza-Klein theories^[4,5]. However, all these proposals are shown to be connected to the existence of solitonic magnetic monopoles^[6-9].

A great obstacle to the development of an electrodynamics with charges and poles is the absence of an adequate lagrangean formulation. This is intimately connected to the difficulty of constructing a regular 4-potential in all space-time. There have been several proposals to circumvent this problem: the introduction of Dirac's string^[10], of the double-valued Wu-Yang potential^[11], of the singular Bollini-Giambiagi potential^[12] and finally the use of non-local wave functions, proposed by Cabbibo and Ferrari^[13]. However, a lagrangean formulation which gives rise to the complete set of

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electromagnetic equations, without necessity of any subsidiary condition, is still lacking.

The main purpose of the present work is to show that a non-local lagrangean can be constructed which gives a correct description of the classical dual electrodynamics provided we postulate the following variational principle: *the dynamics of the system charge-field-monopole is such that the action presents a saddle point which is a minimum with respect to variation of the usual degrees of freedom and a maximum with respect to variation of the dual degrees of freedom.*

Following such prescription we construct the lagrangean density

$$\mathcal{L} = \mathcal{L}_o^e + \mathcal{L}_o^g - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu \mathcal{A}^\mu + g_\mu \tilde{\mathcal{A}}^\mu \quad (1)$$

where j_μ e g_μ are the electric and magnetic 4-currents, respectively. Here we have introduced the Cabibbo-Ferrari generalized field tensor^[13]

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - \epsilon^{\mu\nu\alpha\beta} \partial_\alpha \tilde{A}_\beta \quad (2)$$

and the non-local potentials^[14]

$$\mathcal{A}^\mu = A^\mu + \frac{1}{2} \epsilon^{\mu\gamma\alpha\beta} \int_P^x \partial_\alpha \tilde{A}_\beta d\xi_\gamma \quad (3)$$

$$\tilde{\mathcal{A}}^\mu = \tilde{A}^\mu - \frac{1}{2} \epsilon^{\mu\gamma\alpha\beta} \int_{\tilde{P}}^x \partial_\alpha A_\beta d\xi_\gamma \quad (4)$$

with P and \tilde{P} defined, respectively, by the world lines of the charge and pole.

The first two terms in (1) correspond, respectively, to the free lagrangeans of the electric and magnetic charges, so that the Lagrange's function corresponding to (1) is given by

$$L = L_e + L_g + L_{Maxwell} \quad (5)$$

where

$$L_e = -m (1 - u^2)^{\frac{1}{2}} + e \vec{u} \cdot \vec{\mathcal{A}} - e \mathcal{A}_0 \quad (6)$$

$$L_g = M (1 - v^2)^{\frac{1}{2}} - g \vec{v} \cdot \vec{\tilde{\mathcal{A}}} + g \tilde{\mathcal{A}}_0 \quad (7)$$

$$L_{Maxwell} = -\frac{1}{4} \int d^3x F_{\mu\nu} F^{\mu\nu} \quad (8)$$

Here \vec{u} and \vec{v} stand for the charge and pole velocities, m and M standing for their masses. e and g are their charge strenghts.

In the absence of monopoles we can, using the gauge freedom^[13], set $\tilde{A}^\mu = 0$ and our lagrangean reduces to the usual lagrangean of electromagnetism. Also, in the absence of electric charges, by setting $A^\mu = 0$ we get, apart from an overall sign, the dual lagrangean

$$\mathcal{L} = \mathcal{L}_o - \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - g_\mu \tilde{A}^\mu \quad (9)$$

where

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (10)$$

stands for the dual of the field tensor.

From (2)–(4) and (10) we can show the validity of the relations^[17]

$$F^{\mu\nu} = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu \quad (11)$$

$$\tilde{F}^{\mu\nu} = \partial^\mu \tilde{\mathcal{A}}^\nu - \partial^\nu \tilde{\mathcal{A}}^\mu \quad (12)$$

Variations of the local potentials – for given fixed particles's world lines – lead to variations of the non-local ones. Using (11), (12) and the identity

$$F_{\mu\nu} F^{\mu\nu} = -\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \quad (13)$$

it is a simple matter to check that the extremum condition for the action under such variations leads to Euler-Lagrange equations which correspond to the expected generalized Maxwell's equations

$$\partial_\beta F^{\alpha\beta} = -j^\alpha \quad (14)$$

$$\partial_\beta \tilde{F}^{\alpha\beta} = -g^\alpha \quad (15)$$

Variations with respect to the coordinates of the charge and pole give

$$m \frac{dU^\alpha}{d\tau} = e (\partial^\alpha \mathcal{A}^\beta - \partial^\beta \mathcal{A}^\alpha) U_\beta \quad (16)$$

$$M \frac{dV^\alpha}{d\tau} = g (\partial^\alpha \tilde{\mathcal{A}}^\beta - \partial^\beta \tilde{\mathcal{A}}^\alpha) V_\beta \quad (17)$$

where U^μ and V^μ stand for the 4-velocities of the charge and pole and τ is their proper time. Here, the derivatives of the potentials are taken along the world lines of the particles. Thus, using again (11) and (12), we obtain the correct Lorentz's equations

$$m \frac{dU^\alpha}{d\tau} = e F^{\alpha\beta} U_\beta \quad (18)$$

$$M \frac{dV^\alpha}{d\tau} = g \tilde{F}^{\alpha\beta} V_\beta \quad (19)$$

We can see that the proposed lagrangean, although non-local, leads to all desired local equations of motion. Using the field equations it is possible to show that a change of the paths of integration in (3) and (4) corresponds to a gauge transformation of the non-local potentials. Because this result, it is not necessary to consider variations of these paths to obtain the particles's equations.

It is important to note that (11) and (12) do not imply into the homogeneity of (14) and (15). It is due to the fact that the non-local potentials are not regular, do not obeying the Euler condition. In other words,

$$(\partial^\mu \partial^\nu - \partial^\nu \partial^\mu) \mathcal{A}^\alpha \neq 0 \quad (20)$$

and the same for $\tilde{\mathcal{A}}^\alpha$.

The irregular character of \mathcal{A}^μ , as a function of x and P , is evident once one examines expression (3) in the case of a magnetic monopole at rest in the origin. Any path which goes through the origin turns the integral into a divergent one over a semi-infinite line. Such singularity are essential, since they come from the intersection between the charge's world line and the monopole's one, and are already contained in the equations of motion derived from the lagrangean. In fact, Lorentz's equation (18) allows the charge to come indefinitely close to the monopole, over the line connecting them. But when the superposition occurs, the second member of this equation becomes singular, unless the relative velocity between charge and pole goes to zero. We note however that, whatever the charge's trajectory might be, the singularity of \mathcal{A}^μ will always lie in a 4-hemisphere opposed to that of charge's motion. This discussion is valid also for the dual non-local potential.

Let us consider the dual transformation

$$A^\mu \rightarrow -\tilde{A}^\mu \quad (21)$$

$$\tilde{A}^\mu \rightarrow A^\mu \quad (22)$$

$$j^\mu \rightarrow -g^\mu \quad (23)$$

$$g^\mu \rightarrow j^\mu \quad (24)$$

together with $m \leftrightarrow M$. The lagrangean and action will change sign. As the equations of motion do not depend on the overall sign of these quantities, we can say that the theory remains invariant under such dual transformation.

The sign difference between the free lagrangeans of the electric and magnetic charges (cf. (6) and (7)) may give the impression that the monopole would appear as a particle with negative energy^[15]. This would be of course unacceptable at the classical level. It is possible to see that this is not the case by calculating the total conserved energy of the system charge-field-monopole, using the equations of motion obtained from the lagrangean.

This result apparently contradicts the hamiltonian formulation of the theory. However, the dual simetry and the very form of the lagrangean, obeing a saddle principle, lead us to a hamiltonian formulation which is internally consistent with the theory. In fact, the dual transformation (21)–(24), under which $L_e \rightarrow -L_g$ and $S_e \rightarrow -S_g$, transforms the momentum and hamiltonian of the charge

$$\vec{p}_e \equiv \frac{\partial S_e}{\partial \vec{r}} = \frac{\partial L_e}{\partial \vec{u}} = \frac{m\vec{u}}{(1-u^2)^{\frac{1}{2}}} + e\vec{\mathcal{A}} \quad (25)$$

$$\mathcal{H}_e \equiv -\frac{\partial S_e}{\partial t} = \frac{\partial L_e}{\partial \vec{u}} \cdot \vec{u} - L_e = [m^2 + (\vec{p}_e - e\vec{\mathcal{A}})^2]^{\frac{1}{2}} + e\mathcal{A}_0 \quad (26)$$

into the momentum and hamiltonian of the pole

$$\vec{p}_g = -\frac{\partial S_g}{\partial \vec{r}} = -\frac{\partial L_g}{\partial \vec{v}} = \frac{M\vec{v}}{(1-v^2)^{\frac{1}{2}}} + g\vec{\tilde{\mathcal{A}}} \quad (27)$$

$$\mathcal{H}_g = \frac{\partial S_g}{\partial t} = -\left(\frac{\partial L_g}{\partial \vec{v}} \cdot \vec{v} - L_g\right) = [M^2 + (\vec{p}_g - g\vec{\tilde{\mathcal{A}}})^2]^{\frac{1}{2}} + g\tilde{\mathcal{A}}_0 \quad (28)$$

The above expressions respect the canonical form of Hamilton's equations, since the latter remain invariant under a simultaneous change of sign of \vec{p} and \mathcal{H} . It is also simple to show that (28) corresponds to the correct time evolution generator for the monopole.

We should like to remark that our formulation does not induce any modification for the particles's equations of motion in the gravitational field.

Usually the action for a mass m particle subject to this field is given by

$$S = -m \int ds \quad (29)$$

with

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (30)$$

In the case of a magnetic charge, however, one should consider the action in the form

$$S = M \int ds \quad (31)$$

so that it presents, contrary to (29), a maximum and so that the Lagrange's function reduces to the correct one (cf. (7)) in the absence of the gravitational field.

Since the equations of motion are given by $\delta S = 0$ the sign difference between (29) and (31) will not matter. This is in complete agreement with the Equivalence Principle. Besides, the positive definite character of the energy of the monopole guarantees that it will play the same role as any other particle in what concerns the generation of gravitational field.

In conclusion, we have proposed an action principle which allows for the construction of a non-local classical lagrangean which yields all the equations of electromagnetism with charges and monopoles, without having to resort to additional restrictions or constraints on the dynamics of the particles.

The quantization of the theory remains a challenging open problem. The same can be said of its non-abelian extension. In the same way that magnetic monopoles can be obtained as solitons of non-abelian theories, we can think that electric charges would be given as topological solutions of dual theories to the first^[19–21]. This possibility suggests a unified description of electric and magnetic charges as configurations of bosonic scalar and vector fields. The difficulty lies, however, in the lack of a self-dual lagrangean which contains at the same time the bosonic fields and their respective duals, like

in (1). The introduction of non-local potentials (non-abelian) may be a way to the construction of a saddle point lagrangean formulation.

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- [17] The covariant derivative of the line integral of a tensor $\Lambda^{\mu\dots\nu}$ with respect to the observation point x^μ , extremum of the path of integration P , along this path, is defined as^[18]

$$\partial^\mu \int_P^x \Lambda^{\alpha\dots\nu}(\xi) d\xi_\nu \equiv \lim_{dx_\mu \rightarrow 0} \frac{[\int_{P'} \Lambda^{\alpha\dots\nu}(\xi) d\xi_\nu - \int_P \Lambda^{\alpha\dots\nu}(\xi) d\xi_\nu]}{dx_\mu} \quad (32)$$

where P' is obtained from P by adding to this one an extension dx^μ in the x^μ direction. Thus if the derivation is to be performed in a direction which should be orthogonal to that of the integration, the difference in brackets vanishes and so does the derivative. On the other hand, if we derive in the same direction of the integration, we obtain, according to the Fundamental Theorem of Calculus, $\Lambda^{\alpha\dots\mu}(x)$. Thus we have

$$\partial^\mu \int_P^x \Lambda^{\alpha\dots\nu}(\xi) d\xi_\nu = \Lambda^{\alpha\dots\nu}(x) \delta_\nu^\mu = \Lambda^{\alpha\dots\mu}(x) \quad (33)$$

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